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Motion of relativistic particles in standing wave fields

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Abstract. The motion of relativistic particles in the field produced by two circularly polarised, electromagnetic plane waves travelling in opposite directions is studied. The Klein–Gordon equation is used, i.e. spin effects are neglected, and only motion along the beam direction is considered. If the electromagnetic beams carry opposite polarisation, the particle energy shows a band structure. The transmission of a plane particle wave through a long device is calculated. Transmission occurs only if the particle energy corresponds to an allowed energy band. For other energies the particles are totally reflected. For equal polarisation a similar structure emerges for the particle momentum along the beam direction. The observation of the band structure poses serious experimental problems.

1. Introduction

The propagation of particles in electromagnetic wave fields has found renewed interest in connection with intense laser fields. The most characteristic feature of particles propagating in monochromatic plane wave fields is the concept of quasi-levels. If p is the energy–momentum vector of the particle outside the field and k the (four-) wave vector of the field, the wavefunction is a superposition of plane waves with four-momenta

$$p_n = p_{\text{eff}} + nk, \qquad p_{\text{eff}} = p + (ea)^2 k/2pk, \qquad n = 0, \pm 1, \pm 2, \dots \tag{1}$$

p_{eff}^2 is a constant (the square of an effective mass) which depends on the mass of the particle as well as the intensity and polarisation of the field. All effects due to the influence of the external field can be understood in terms of the intensity-dependent effective mass and the quasi-level structure. Despite the simplicity of the concept, little has been achieved with respect to an experimental verification of its consequences. The main reason is the fact that the order of magnitude of most effects is controlled by the parameter

$$\nu = ea/mc^2,$$

related to the charge e and mass m of the particle and the maximal amplitude a of the wave. For electrons we have, in terms of the irradiation density S (in W cm^{-2}) and the wavelength λ_L (in cm) of the laser,

$$\nu^2 \simeq 7{\cdot}5 \times 10^{-11} \lambda_L^2 S,$$

and this is only appreciable for extremely strong lasers. In addition, one has to observe that the effective mass is essentially of classical origin, so that the quantum nature of the particle does not enter vitally. It therefore seems worthwhile to look for different field configurations in which a genuine quantum effect could also emerge (Meckbach 1976).

It was noted some time ago by Berson and Valdmanis (1973) that the Klein–Gordon equation for a particle in an external wave field can be solved exactly in a special situation. The field has to be made up of two circularly polarised, monochromatic plane waves of equal frequency propagating in opposite directions (so that a standing wave is produced), and the particle (three-) momentum has to be parallel to the wave vector. The physical consequences have, however, not been pursued very far. Especially, almost no quantitative consequences have been derived. We shall therefore take up this problem again. In contrast to the work of Berson and Valdmanis (1973), where a quantised electromagnetic field is also considered, we shall treat the standing wave as a prescribed classical field, which is an excellent approximation for high intensities. The two possibilities of opposite/equal polarisation of the waves forming the external field shall be treated separately, since the qualitative and quantitative consequences are entirely different.

2. Opposite polarisation

The Klein–Gordon equation for a spinless particle in an external field A^μ reads

$$[(\mathrm{i}\,\partial^\mu+\epsilon A^\mu)(\mathrm{i}\,\partial_\mu+\epsilon A_\mu)-\kappa^2]\phi=0. \tag{2}$$

Here $\epsilon=e/\hbar c$ is related to the charge, and $\kappa=mc/\hbar=2\pi/\lambda_{\mathrm{C}}$ to the mass/Compton wavelength (λ_{C}) of the particle.

The particle current is

$$j^\mu=\mathrm{i}\phi^*\overleftrightarrow{\partial^\mu}\phi+2\epsilon A^\mu\phi^*\phi, \tag{3}$$

where we have used the symbol

$$a\,\overleftrightarrow{\partial^\mu}b=a\,\partial^\mu b-b\,\partial^\mu a. \tag{4}$$

As a consequence of equation (2) the current must be conserved,

$$\partial_\mu j^\mu=0. \tag{5}$$

If the waves producing the field are propagating in the $\pm z$ direction, the vector potential reads

$$A^\mu=2a\cos\omega z\,(0,\cos\omega x_0,\ -\sin\omega x_0,\ 0), \tag{6}$$

where ω is the frequency and a the intensity parameter. The vector potential (6) is invariant under a time translation combined with an appropriate rotation,

$$A^k(x_0,\boldsymbol{x})=D^{kl}(\omega b)A^l(x_0+b,\boldsymbol{x}), \tag{7}$$

where $D^{kl}(\alpha)$ describes a rotation about the z axis by an angle α, and b is an arbitrary constant. This symmetry is all that remains here from the larger class of symmetry transformations advocated by Richard (1972) for a plane wave with circular polarisation. As in other problems, the symmetry has consequences for the wavefunction. If the momentum of the particle is in the z direction, it is invariant with respect to the rotation

† The invariance expressed in equation (7) also allows for a solution of the Dirac equation in this case. This will eventually be published in a separate paper.

D^{kl}. Hence we have time translation invariance, and the dependence on time can be separated.

In what follows we shall consider this simple situation. Since the potential (6) does not depend on x and y, we may split off an exponential

$$\phi(x) = \psi(x_0, z) \exp[i(xp_x + yp_y)], \tag{8}$$

with constant p_x, p_y. If we assume

$$p_x = p_y = 0, \tag{9}$$

the remaining differential equation for ψ is invariant under time translations (thus energy is conserved) and can be separated. With the *ansatz*

$$\psi(x_0, z) = \phi(\xi) \exp(-ix_0 p_0), \tag{10}$$

where

$$\xi = \omega z, \qquad p_0 = E/\hbar c, \tag{11}$$

we obtain the Mathieu equation

$$(d/d\xi^2 + \Lambda - 2l^2 \cos 2\xi)\phi(\xi) = 0, \tag{12}$$

with the parameters

$$l = \epsilon a/\omega = \nu \lambda_L/\lambda_C \tag{13}$$

$$\Lambda = (p_0^2 - \kappa^2)/\omega^2 - 2l^2. \tag{14}$$

This equation would result from a one-dimensional Schrödinger problem with an 'effective potential' $\sim \boldsymbol{A}^2$ and an 'energy' $\sim p_0^2 - \kappa^2$.

Before we discuss the solutions, we shall write down the components of the current density (3) in terms of ϕ. We have

$$j_0 = 2p_0\phi^*\phi, \qquad j_z = -i\omega\phi^*(\overleftrightarrow{d/d\xi})\phi,$$

$$(j_x, j_y) = 4\epsilon a\phi^*\phi \cos \xi(\cos \omega x_0, -\sin \omega x_0), \tag{15}$$

and the continuity equation reduces to

$$(d/d\xi)j_z = 0. \tag{16}$$

The standard theory of Mathieu's equation (see e.g. Meixner and Schäfke 1954) tells us that

$$\phi_\pm(\xi) = P(\pm\xi) \exp(\pm i\tau\xi) \tag{17}$$

form a set of fundamental solutions of equation (11). Here P is a periodic function,

$$P(\xi + \pi) = P(\xi),$$

and the characteristic exponent $\tau(\Lambda, l^2)$ is a constant. The solutions (17) are linearly independent, unless τ is an integer. If ϕ is expanded in a Fourier series,

$$\phi_\pm(\xi) =: me_{\pm\tau}(\xi, \Lambda) = \sum_{r=-\infty}^{\infty} C_{2r}(\Lambda, l^2) \exp[i\xi(2r + \tau)], \tag{18}$$

the coefficients satisfy a three-term recurrence relation. Trying a physical interpretation, we could say that each term in (18) corresponds to a quasi-momentum on the reciprocal lattice,

$$p_z(r) = \pm\omega(2r+\tau), \qquad r = 0, \pm 1, \pm 2, \ldots. \tag{19}$$

Inserting the solutions (17) into (15), we observe that j_z can only be constant if either (a) τ is complex and $j_z = 0$, or (b) τ is real.

In case (a) the solutions are called unstable and are unbounded for large positive or negative values of ξ. j_0 and $j_{x,y}$ are unbounded, and the solutions cannot be normalised. The charge contained in a finite interval in ξ would increase exponentially with the length of that interval. In addition, the quasi-momentum (19) becomes complex. If we require the total charge (which is constant in time) to be finite, we must rule out the unstable solutions. In case (b) the solutions are called stable and are physically meaningful, since ϕ and j^μ are bounded and the quasi-momentum is real.

Concrete values of $\tau = \tau(\Lambda, l^2)$ must be calculated from equation (12). For given values of l^2, stable solutions exist only within certain domains of Λ. Because of relation (14), this means that the energy shows a band structure as for particles in solids. In the Λ, l^2 plane the stable regions are situated between the characteristic curves $\Lambda = a_n(l^2)$ and $\Lambda = b_{n+1}(l^2)$ numbered by an integer $n = 0, 1, 2, \ldots$. This represents the so-called stadility chart. At these curves the characteristic exponent assumes the values n and $n+1$ respectively. The curves can be determined by numerical analysis (see Meixner and Schäfke 1954). The axis $l^2 = 0$ is a stable region for any $\Lambda > 0$, as can also be seen directly from the differential equation. At sufficiently high energies, i.e. for

$$p_0^2 - \kappa^2 \gg l^2\omega^2,$$

the periodic term in equation (12) is a small perturbation, and the solution will remain stable (i.e. the energy spectrum remains continuous) almost everywhere. In order to discover the band structure we have therefore to consider moderate energies or very high intensities. From the stability chart we see that the stable regions become more and more narrow for increasing l^2 (for fixed values of Λ). For large values of this parameter the width of the stable region decreases exponentially, and we obtain practically discrete energy eigenvalues numbered by n. It has to be observed that the asymptotic domain $l^2 \gg \Lambda$ can be reached experimentally with available intensities ν (compare (13)). Then we can use the asymptotic expansion valid for large l^2,

$$\Lambda \simeq -2l^2 + 2(2n+1)l - \tfrac{1}{4}(2n^2+2n+1) - (1/32l)[2n^3+3n^2+3n+1] + \ldots. \tag{20}$$

We shall take into account only the first two terms: the next term contributes less than 1% for $l \geqslant 100$ and $n < l/10$. Then we obtain the energy formula

$$E_n \simeq mc^2[1+2(2n+1)\Delta]^{1/2} \simeq mc^2[1+(2n+1)\Delta], \tag{21}$$

where the splitting constant Δ is given by

$$\Delta = l\omega^2/\kappa^2 = \nu\lambda_C/\lambda_L. \tag{22}$$

Thus the level distance does not, in fact, depend on n and becomes very small in practice. (Note, however, that the level (or band) structure is a genuine quantum phenomenon.)

On the characteristic curves a_n and b_{n+1} a fundamental system of solutions of equation (12) consists of the periodic Mathieu functions $\mathrm{ce}_n(\xi/l^2)$ and $\mathrm{se}_{n+1}(\xi/l^2)$ respectively, and two other linearly increasing non-periodic solutions. For a given value of n, the periodic functions (with period π or 2π) can be used as eigenfunctions corresponding to E_n for large l^2. These are

$$\mathrm{ce}_n(\xi, l^2) = F_n + O(l^{-3/4}), \tag{23}$$

$$\mathrm{se}_{n+1}(\xi, l^2) = \sin\xi F_n + O(l^{-3/4}), \tag{24}$$

where F is given in terms of parabolic cylinder functions

$$F_n = (\pi l/2)^{1/4}(n!)^{-1/2} D_n(2\sqrt{l}\cos\xi), \tag{25}$$

$$D_n(x) = (-1)^n\, e^{x^2/4}(d/dx)^n\, e^{-x^2/2} = 2^{-n/2}\, e^{-x^2/4} H_n(x/\sqrt{2}), \tag{26}$$

which are the eigenfunctions of the harmonic oscillator. The physics behind this result is easy to understand in terms of the analogy with a Schrödinger problem as mentioned before. The effective potential $\boldsymbol{A}^2$ has maxima at $\xi = 0, \pi, 2\pi, \ldots$ and minima at $\xi = \pi/2, 3\pi/2, \ldots$, where it assumes the value zero. For large l^2 these maxima become very large, so that the particle density is concentrated close to the minima of the potential (this can be seen by computing j^0 with the asymptotic eigenfunctions), where the potential may be approximated by an oscillator potential. We note that in this case the two eigenfunctions (23) and (24) are nearly identical apart from the fact that se_{n+1} and ce_n have the opposite sign in every second minimum of the potential. The z component of the current vanishes asymptotically. The transverse components oscillate in time and are concentrated close to the minima of the effective potential, but vanish at these points because of the factor $\cos\xi$.

The band structure of the energy emerges for smaller values of l. In order to obtain a qualitative picture we may, as in solid state physics, define a line in the Λ, l^2 plane, which marks the 'middle' between the asymptotic domain (l^2 large) and the region where the particle is almost free (l^2 small), by

$$\Lambda = 2l^2, \qquad \text{i.e. } E_{\text{th}} = mc^2(1+4\nu^2)^{1/2}. \tag{27}$$

This line marks the classical threshold for transmission of particles (Latal 1977). Close to this curve we have energy bands. The corresponding eigenfunctions can be represented in terms of Wannier functions, which can be constructed as in solid state physics.

Putting in numbers, it becomes clear that the band structure shows up at small energies in general. If the threshold corresponds to a kinetic energy $E_{\text{th}} - mc^2$ of 1 keV for electrons, we need $\nu^2 \sim 10^{-3}$ or $\lambda_L^2 S \sim 1{\cdot}3 \times 10^7$. With the strongest infrared lasers constructed for fusion research one may obtain the corresponding irradiation densities; however, it seems hardly possible to manage a clean standing wave with these lasers. Perhaps one should consider microwave devices in this context, although the large value of $\lambda_L^2 S$ presents problems for these also. In any case, it has to be noted that the corresponding value of l^2 comes out very large.

Close to the threshold we cannot use the asymptotic formulae (20) or (23) and (24), since in this domain both Λ (or n) and l^2 are large. Asymptotic formulae which are valid in this region can be found in a very general investigation by Langer (1934). The characteristic curves in the vicinity of the classical threshold are given by

$$a_n(l^2) = 2l^2 + 4l\{(n+\tfrac{1}{4})\pi - 4l\}/(\log 64l + C + \pi/2) + O(1), \tag{28}$$

$$b_n(l^2) = 2l^2 + 4l\{(n-\tfrac{1}{4})\pi - 4l\}/(\log 64l + C - \pi/2) + O(1), \tag{29}$$

where C denotes Euler's constant. If a point on the straight line $\Lambda = 2l^2$, which marks the classical threshold, happens to lie in a stable band $a_n(l^2) \leq \Lambda \leq b_{n+1}(l^2)$, the corresponding band number is given by

$$n = [4l/\pi - \tfrac{1}{4}], \tag{30a}$$

while if it comes to lie in an unstable band $b_n(l^2) \leq \Lambda \leq a_n(l^2)$, it is given by

$$n = [4l/\pi + \tfrac{1}{4}], \tag{30b}$$

with $[x]$ denoting the largest integer smaller than x. The width of the nth stable band is

$$\Delta_n \Lambda \equiv b_{n+1}(l^2) - a_n(l^2) = 2l\pi\{L + (2n+1)\pi - 8l\}/(L^2 - \pi^2/4), \tag{31}$$

and the width of the nth unstable band is

$$\delta_n \Lambda \equiv a_n(l^2) - b_n(l^2) = 2l\pi\{L - (2n\pi - 8l)\}/(L^2 - \pi^2/4), \tag{32}$$

with $L = \log 64l + C$. Note that $\Delta_n \Lambda$ and $\delta_u \Lambda$ are roughly proportional to $l/\log l$ for large l, since the last term in the numerator is small due to equations (30a, b). In contrast to the widths of the bands, the distances between the lower or upper edges of the energy bands are independent of n,

$$a_{n+1}(l^2) - a_n(l^2) = 4l\pi/(L + \pi/2), \qquad b_{n+1}(l^2) - b_n(l^2) = 4l\pi/(L - \pi/2). \tag{33}$$

The consequences of equations (31)–(33) are illustrated in figure 1, if one observes that $\Delta p \approx \Delta\Lambda/4l$ for constant l^2 near the classical threshold (cf equation (43)).

If, on the other hand, the threshold is at very small energies (which will happen for lasers with realistic irradiation densities), one has to work far above threshold. In this case, one can use perturbation expansions in powers of l^2 even for large l^2, as long as l^2/τ is small enough. We have

$$\Lambda \simeq \tau^2(1 + l^4/2\tau^4 + O(l^8/\tau^8)), \tag{34}$$

valid for any (integer or non-integer) $\tau \gg 1$. The corresponding expansion for the solution reads (for non-integer τ)

$$me_\tau \simeq e^{i\tau\xi}\Big[1 - \frac{l^2}{\tau}\frac{i}{2}\sin 2\xi + \frac{l^2}{2\tau^2}\Big(\cos 2\xi + \frac{l^2}{8}(\cos 4\xi - 1)\Big) + O\Big(\frac{l^3}{\tau^3}\Big)\Big]. \tag{35}$$

The width of the nth stable band ($n \gg 1$) is

$$\Delta_n \Lambda \simeq 2n + 1 - l^4/n^3 + O(l^8), \tag{36}$$

whereas the width of the nth qble zone decreases as

$$\delta_n \Lambda \simeq O(l^{2n}/n^{n-1}). \tag{37}$$

3. Transmission and reflection

In order to understand the physical aspects of our solution, we shall now consider the penetration of a plane particle wave through the standing wave field. The corresponding classical problem has been treated before by Latal (1977). General features of wave propagation in a periodic, continuous medium are discussed in an excellent book by Brillouin (1946) and also apply here. Thus we can expect that the field configuration shows transmission bands if the energy corresponds to a stable zone. These bands should become broad for energies above the classical threshold (27) and narrow below this value.

For the calculation we shall assume that the field is present in a domain

$$-(m+\tfrac{1}{2})\pi \leq \xi \leq (m+\tfrac{1}{2})\pi \tag{38}$$

(here m is a large integer) and vanishes outside this region. The wavefunction in the interior region is given by a superposition of Mathieu functions

$$\phi(\xi) = Dme_\tau(\xi) + Eme_\tau(-\xi), \qquad \tau > 0. \tag{39}$$

Outside this region we assume

$$\phi(\xi) = A\,\mathrm{e}^{\mathrm{i}p\xi} + B\,\mathrm{e}^{-\mathrm{i}p\xi}, \qquad \xi \leq -(m+\tfrac{1}{2})\pi, \tag{40}$$

$$\phi(\xi) = F\,\mathrm{e}^{\mathrm{i}p\xi}, \qquad \xi \geq (m+\tfrac{1}{2})\pi. \tag{41}$$

Since the energy is conserved, we have, because of equation (14),

$$p^2 = (p_0^2 - \kappa^2)/\omega^2 = \Lambda + 2l^2. \tag{42}$$

Thus a given value of p fixes the characteristic exponent τ. The constants A, B, D, E, F are related by the boundary conditions for ϕ and ϕ' at the boundaries of the region (where the field vanishes). The transmission and reflection coefficients of the device are defined as usual by

$$T = F^*F/A^*A, \qquad R = B^*B/A^*A. \tag{43}$$

Computing the z component of the current (15), which is constant because of equation (16), we obtain

$$\omega^{-1}j_z = 2p(A^*A - B^*B) = 2pF^*F.$$

Thus we have

$$T + R = 1.$$

If now the momentum p of the incoming particle corresponds to an unstable region, the current will vanish for an infinite domain (for a finite region we obtain in fact exponential damping with m), and we have

$$T = 0, \qquad R = 1, \tag{44}$$

i.e. the particle wave is totally reflected. For the same reason we obtain this result also for very large values of l^2 (asymptotically).

If the momentum corresponds to a stable region, transmission is possible. Using the periodicity properties of Mathieu functions, we obtain from the boundary conditions

$$T = 1 - R = (1-G^2)^2/[1 + G^4 - 2G^2\cos 4\pi\tau(m+\tfrac{1}{2})], \tag{45}$$

where

$$G = (pK(\tau) - N(\tau))/(pK(\tau) + N(\tau)) \tag{46}$$

is given in terms of Mathieu functions at $\xi = \pi/2$,

$$K(\tau) = \sum_n (-1)^n C_{2n} = me_\tau\left(\frac{\pi}{2}\right)\exp\left(\frac{-\mathrm{i}\pi\tau}{2}\right)$$

$$N(\tau) = \sum_n (-1)^n(\tau+2n)C_{2n} = -\mathrm{i}me'_\tau\left(\frac{\pi}{2}\right)\exp\left(\frac{-\mathrm{i}\pi\tau}{2}\right). \tag{47}$$

We note that the transmission coefficient oscillates rapidly between the values

$$[(1-G^2)/(1+G^2)]^2 \leqslant T \leqslant 1.$$

The average value is

$$\langle T\rangle = (1-G^2)^2/(1+G^4). \qquad (48)$$

At the band edges we have $T = 1$, but $\langle T\rangle = 0$.

In order to see what happens, we have computed $\langle T\rangle$ as a function of p for given values of l^2 near the classical threshold. The values of Λ and l^2 are such that the doubly asymptotic formulae of Langer (1934) had to be employed. The results are shown in figure 1, where the arrow marks the classical threshold. The values of $l^2 = 0{\cdot}163\times10^4$, $\times10^6$, $\times10^8$ correspond to kinetic energies of $0{\cdot}98\times10^{-2}$, $\times1$, $\times10^2$ eV for $\lambda_L = 1$ μm and $0{\cdot}98\times10^{-4}$, $\times10^{-2}$, $\times1$ eV for $\lambda_L = 10$ μm. The pattern mentioned at the beginning of this section is evident from figure 1.

From equation (33) we know that the separation Δp of neighbouring allowed zones decreases slowly as $(\log l)^{-1}$. This fact may pose severe restrictions on the monochromaticity of the electrons to be used in experiments. A quantitative analysis would

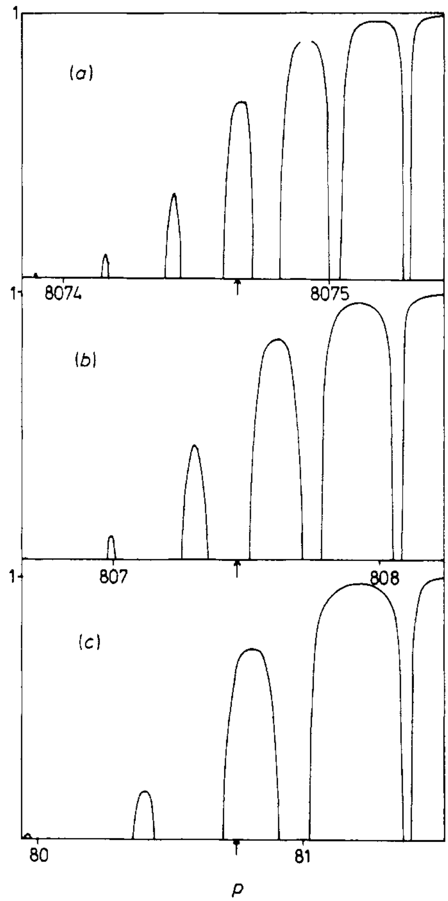

Figure 1. Average transmission coefficient as a function of p for different values of l^2, (*a*) $1{\cdot}63\times10^7$, (*b*) $1{\cdot}63\times10^5$, (*c*) $1{\cdot}63\times10^3$. The arrow marks the classical threshold.

require the use of wave packets allowing for an energy spread of the particles. We note, finally, that along the classical threshold, using Langer's formulae, the average transmission coefficient can be shown to fulfil $\langle T\rangle \leq \frac{2}{3}$. Figure 2 exhibits the decrease of the average transmission coefficient for constant τ as a function of l^2.

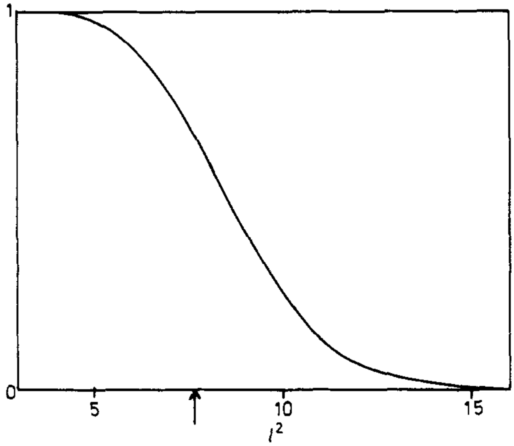

Figure 2. Average transmission coefficient as function of l^2 for $\tau = 3{\cdot}6$. The arrow marks the classical threshold.

Above threshold the width of the transmission bands increases rapidly. Thus the reflection zones become narrow and well separated from each other, so that an energy spread of the incoming electrons should not affect the spectrum of the reflected particles. At energies very far above threshold we can compute the transmission coefficient using (35). The result is

$$\langle T\rangle = 1-\langle R\rangle \simeq 1-2(l/\tau)^4, \tag{49}$$

which gives very small values for the reflection coefficient in realistic cases. To see this, we have only to observe that, within our approximation,

$$4l^2/\tau^2 \simeq 4l^2/(\Lambda+2l^2) = [E_{\rm th}^2-(mc^2)^2]/[E^2-(mc^2)^2],$$

which is for non-relativistic particles the ratio of the kinetic energy at threshold to the actual kinetic energy.

4. Equal polarisation

In this case the vector potential becomes

$$A^\mu = 2a\cos\omega x_0(0, \cos\omega z, \sin\omega z, 0) \tag{50}$$

and is invariant under a space translation combined with an appropriate rotation,

$$A^k(x^0, z) = D^{kl}(-\omega b)A^l(x^0, z+b), \tag{51}$$

where D^{kl} is the same rotation matrix as in equation (7).

Performing the same steps (8), (9) as before we obtain an equation which is invariant under translations in z directions. Thus we put

$$\psi(x^0, z) = \pi(\mu)\exp(izp_z), \tag{52}$$

where now

$$\eta = \omega x_0 + \pi/2, \tag{53}$$

and obtain the Mathieu equation

$$(d^2/d\eta^2 + \Sigma - 2l^2 \cos 2\eta)\pi(\eta) = 0, \tag{54}$$

with l^2 given by equation (13) and

$$\Sigma = (p_z^2 + \kappa^2)/\omega^2 + 2l^2. \tag{55}$$

The further steps can be carried through formally as before. Now the individual terms in the Fourier series can be interpreted to have quasi-energies

$$p_0(r) = \pm(2r + \tau)\omega \tag{56}$$

rather than quasi-momenta, since the variable η is proportional to the time coordinate. For the stability consideration we have to observe that now the roles of j_0 and j_z are exchanged, since the continuity equation becomes

$$(d/d\eta)j_0 = 0. \tag{57}$$

In the unstable case we have now $j^0 = 0$: the norm (and the charge density) of the unstable solutions vanishes. Since j^0 is not positive definite in the Klein–Gordon theory, we cannot draw the conclusion that π must vanish, however. The other components of the current increase exponentially for large (positive or negative) values of η, as does π, and the quasi-energies (56) are complex. The total charge of these solutions is zero; obviously they describe pair production from the vacuum. If we want to have a one-particle situation, we must rule them out. For the stable solutions, π and j^μ is bounded with respect to η and the quasi-energies are real. Practically, however, there is little chance to observe a band-type structure for the momentum p_z and, correspondingly, pair production from the vacuum between the bands.

This can be shown as follows. Since p_z^2 has to be positive, we obtain from (55)

$$\Sigma > (\lambda_L/\lambda_C)^2 + 2l^2. \tag{58}$$

The asymptotic formula (20) cannot be used for Σ since this would violate (58). For optical devices the first term on the r.h.s. is very large. Even for appreciable values of l^2 we have therefore $\Sigma \gg l^2$, and this means that we have in practice a continuous spectrum. This situation could be changed only if the second term in (58) was no longer small compared with the first one. Because of (13) this can happen only for extremely high intensities. We shall therefore not pursue this case further.

5. Conclusions

It has been shown that a band structure emerges if particles move in the field of a standing wave. This structure refers to the particle energy if the fields making up the standing wave carry opposite polarisation, whereas we have momentum bands for equal polarisation. Observation of the energy bands is very hard in direct experiments with particle beams, because the classical threshold for transmission lies at very low energies for realistic intensities and wavelengths. The momentum bands can hardly be seen at all, since the distance between the bands is too small. We emphasise, finally, that the existence and the shape of the energy and momentum bands are genuine intensity-dependent quantum effects, which do not have manifest counterparts in the plane wave case.

References

Berson I and Valdmanis J 1973 *J. Math. Phys.* **14** 1481–4
Brillouin L 1946 *Wave Propagation in Periodic Structures* (New York: McGraw-Hill)
Langer R E 1934 *Am. Math. Soc. Trans.* **36** 637–95
Latal H 1977 *Phys. Lett.* A **64** 175–6
Meckbach R 1976 *Diploma Thesis* University of Tübingen
Meixner J and Schäfke F W 1954 *Mathieu'sche Funktionen und Sphäroidfunktionen* (Berlin: Springer)
Richard J L 1972 *Nuovo Cim.* A **8** 485–500